MAC 2312 — Calculus II

Guided Notes

Section 11.9

Representation of functions as a power series

goals of this section:

① Creating/recognizing a_n from expanded power series.

② Derivatives

③ Integrals

Example 1

Find a function f that is represented by the power series $\sum_{n=0}^{\infty} (-1)^n x^n$

if $|x|<1$, then the series is geometric

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

and it converges

$a = 1$

$ar = -x$

so $r = -x$

$$\text{sum} = \frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

so

$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$	$\|x\| < 1$

Example 2

Find a power series representation for $\frac{1}{(1+x)^2}$

we just showed that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

let's expand on the result

all with $|x| < 1$

so $f(x) = (1+x)^{-1} = \frac{1}{x+1}$

$f'(x) = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$

so we need to take the derivative of each term of $\sum (-1)^n x^n$ but then change the sign.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

$$\frac{-1}{(1+x)^2} = \left[0 - 1 + 2x - 3x^2 + \cdots + (-1)^n \cdot n x^{n-1} + \cdots\right]$$

we want $\frac{1}{(1+x)^2}$, so we need to multiply by -1

$$\boxed{\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \cdots + (-1)^{n+1} n x^{n-1} + \cdots}$$

Example 3

Find a power series representation for
$\ln(1+x)$ if $|x|<1$

For this one, we will use an integral.

To initially make that decision, think about

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

or some other convergent series that we know.

Then decide if what you need to find might be a derivative or an integral of the original function.

In this case $\frac{d}{dx}\left(\ln(1+x)\right) = \frac{1}{1+x}$

so $\ln(1+x) = \int \frac{1}{1+x}\,dx$

so we start with

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$$

with

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n (t)^n = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots$$

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$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots\right)dt$$

$$= \int_0^x 1\,dt - \int_0^x t\,dt + \int_0^x t^2\,dt + \cdots + \int_0^x (-1)^n t^n\,dt + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots$$

$|x| < 1$

so $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad \text{for } |x| < 1$$

Example 4

Calculate to $\ln(1.1)$ to five decimal places using the results from Example 3.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots$$

$$\ln(1.1) = \ln(1+0.1) \qquad |x|<1$$

so $x = 0.1$

How can we tell how many terms we need to be accurate to five decimal places?

Answer: We use this theorem

If $\sum(-1)^{n-1} a_n$ is an alternating series such that $a_k > a_{k+1} > 0$ for every positive integer k and if $\lim_{n\to\infty} a_n = 0$, then the error involved in approximating the sum S by the nth partial sum S_n is numerically less than a_{n+1}

$$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5} - \cdots$$

$$= 0.1 - 0.0005 + 0.000333 - 0.000025 + \underbrace{0.00000}_{5\text{ zeros}}2 -$$

add the first 4

$$\cong \boxed{0.09531}$$

$\boxed{\text{Example 5}}$

Find a power series representation for arctan x

observe that $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$

or $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$

$\boxed{\text{so}}$

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt$$

Recall that $\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$ geometric with $|x|<1$

So let's make some adjustments: replace x with x^2

$$\frac{1}{1+(x^2)} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^{2n})$$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + \cdots$$

now we integrate each piece.

$$\tan^{-1}(x) = \int_0^x \left(1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + \cdots\right) dt$$

$$= t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots (-1)^n \frac{t^{2n+1}}{2n+1} + \cdots \Bigg|_{t=0}^{t=x}$$

so

$$\boxed{\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \qquad |x|<1}$$

Just as with any power series, we can check the endpoints to see if the series also converges. For this example, $|x| = 1$ also works.

Example of a proof

Example 6

Prove that e^x has the power series representation

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

lets look at $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

This powerseries is absolutely convergent for all x values.

so $f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

so $f'(x) = f(x)$ for every x

so we suspect this is true, but we need proof.

Now we need to use a theorem involving exponents and logarithms. It says

$$\boxed{\text{If } q'(t) = c\,q(t), \text{ then } q(t) = q(0)e^{ct}}$$

so for us $q = f$, $t = x$ and $c = 1$ so

$f(0) = 1 + 0 + \frac{0!}{2} + \cdots = 1$

$f(x) = (1)e^x$

$$\boxed{f(x) = e^x}$$

Example 7

approximate $\int_0^{0.1} e^{-x^2}\,dx$

accurate to five decimal places

Officially, the x's must be replaced with t's in the new integral

Let's use our results from example 6.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

let $x = -t^2$

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots$$

$$\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1} e^{-t^2}\,dt$$

$$= \int_0^{0.1} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots\right) dt$$

$$= \left(t - \frac{t^3}{3} + \frac{t^5}{5(2!)} - \frac{t^7}{7(3!)} + \cdots \frac{(-1)^n t^{2n+1}}{n!\,(2n+1)} + \cdots\right)\Bigg|_{t=0}^{t=0.1}$$

$$= 0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{10} - \frac{(0.1)^7}{42} + \cdots$$

$$= 0.1 - 0.0003333 + 0.\underbrace{00000}_{\text{5 zeros so I can stop}}1$$

$$\cong \boxed{0.09967}$$

Section 11.9 | Practice Problems

Here are some for you to try
(Hints about the solutions are on the next page. Try them on your own first.)

Find the power series representation for $f(x)$ and specify the radius of convergence.

1. $f(x) = \dfrac{1}{1-4x}$

2. $f(x) = \dfrac{x}{2-3x}$

3. $f(x) = \dfrac{x^2}{1-x^2}$

Recall that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$

4. Find $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

5. Find xe^{3x}

Use Infinite series to approximate each of following (accurate to 4 decimal places)

6. $\displaystyle\int_0^{1/2} \arctan(x^2)\,dx$ 7. $\displaystyle\int_0^{1/3} \frac{1}{1+x^6}\,dx$

8. Find the power series representation for $f(x) = \displaystyle\int_0^x \ln(1+t^2)\,dt$

11.9 solution hints

1. Replace x with $-4x$ in $\frac{1}{1+x}$

so $\sum_{n=0}^{\infty} \frac{1}{1-4x} = \sum_{n=0}^{\infty} (4x)^n = \sum_{n=0}^{\infty} 4^n x^n$ $\quad |4x|<1$ or $|x|<\frac{1}{4}$

2. $\frac{x}{2-3x} = \frac{x}{2}\left[\frac{1}{1-(\frac{3}{2}x)}\right]$

$= \frac{x}{2}\sum_{n=0}^{\infty} (\frac{3}{2}x)^n = \sum_{n=0}^{\infty} \frac{3^n x^{n+1}}{2^{n+1}}$ $\quad |\frac{3}{2}x|<1$ or $|x|<\frac{2}{3}$

geometric with $r=(\frac{3}{2}x)$

3. $\frac{x^2}{1-x^2} = x^2\left[\underbrace{\frac{1}{1-x^2}}_{\text{geometric}}\right]$ $\quad \boxed{\begin{array}{l} a=x^2 \\ r=x^2 \end{array}}$

$= x^2\left[1+x^2+x^4+\cdots+\right] = x^2+x^4+\cdots$

$= \sum x^{2n+2} \quad |x^2|<1 \Rightarrow |x|<1$

4. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!}$ $\quad$ term by term

final result $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

5 replace x with 3x then multiply results by x

$$xe^{3x} = x\sum_{n=0}^{\infty}\left(\frac{3^n}{n!}\right)x^n = \sum_{n=0}^{\infty}\left(\frac{3^n}{n!}\right)x^{n+1}$$

6 use $\tan^{-1}x$ we found in example 5

$$\tan^{-1}(x)\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

replace x with x^2

$$\int \tan^{-1}(x)\,dx = \frac{x^3}{3} - \frac{x^7}{3\cdot 7} + \frac{x^{11}}{5\cdot 11} - \frac{x^{15}}{7\cdot 15} + \cdots$$

two terms will be enough

$0.041666 - 0.00037202 \cong \boxed{0.0413}$

7 start with $\frac{1}{1+x}$ replace x with x^6

$$\frac{1}{1+x^6} = 1 - x^6 + x^{12} - \cdots \qquad |x|<1$$

need 2 terms

Convergent alternating series.

$$\int \frac{1}{1+x^6}dx = x - \frac{x^7}{7} + \frac{x^{13}}{13} - \cdots \Big|_0^{1/3}$$

$\boxed{0.3333}$

$\cong 0.33333 - 0.000007$

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8 use example 3 that finds $\ln(1+x)$ Let $x = x^2$

final answer

$$\sum_{n=1}^{\infty} \int_0^x (-1)^{n+1} \frac{t^{2n}}{n} \, dt$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)n}$$